Maximum norms of graphs and matrices, and their complements

Vladimir Nikiforov [*][†] and Xiying Yuan [‡][§]

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Abstract

Given a graph G, let $\left\Vert G\right\Vert_*$ denote the trace norm of its adjacency matrix, also known as the *energy* of G. The main result of this paper states that if G is a graph of order n, then

$$\left\Vert G\right\Vert_* + \left\Vert \overline{G}\right\Vert_* \leq (n-1)\left(1+\sqrt{n}\right),$$

where $\overline{G}$ is the complement of G. Equality is possible if and only if G is a strongly regular graph with parameters $(n,(n-1)/2,(n-5)/4,(n-1)/4)$, known also as a conference graph.

In fact, the above problem is stated and solved in a more general setup - for nonnegative matrices with bounded entries. In particular, this study exhibits analytical matrix functions attaining maxima on matrices with rigid and complex combinatorial structure.

In the last section the same questions are studied for Ky Fan norms. Possibe directions for further research are outlined, as it turns out that the above problems are just a tip of a larger multidimensional research area.

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1 Introduction and main results

In this paper we study the maxima of certain norms of nonnegative matrices with bounded entries. We shall focus first on the trace norm $\left\Vert A\right\Vert_*$ of a matrix A, which is just the sum of

[*]Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA; email: *vnikifrv@memphis.edu*

[†]Research supported by NSF grant DMS-0906634.

[‡]*Corresponding author.* Department of Mathematics, Shanghai University, Shanghai, 200444, China; email: *xiyingyuan2007@hotmail.com*

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its singular values. The trace norm of the adjacency matrix of graphs has been intensively studied recently under the name *graph energy.* This research has been initiated by Gutman in [3]; the reader is referred to [6] for a comprehensive recent survey and references.

One of the most intriguing problems in this area is to determine which graphs with given number of vertices have maximal energy. A cornerstone result of Koolen and Moulton [5] shows that if G is a graph of order n, then the trace norm of its adjacency matrix $\|G\|_*$ satisfies the inequality

$$\|G\|_* \leq \left(1+\sqrt{n}\right)\frac{n}{2}, \tag{1}$$

with equality holding precisely when G is a strongly regular graph with parameters

$$\left(n, \left(n+\sqrt{n}\right)/2, \left(n+2\sqrt{n}\right)/4, \left(n+2\sqrt{n}\right)/4\right).$$

Note that G can be characterized also as a regular graph of degree $\left(n+\sqrt{n}\right)/2$, whose singular values, other than the largest one, are equal to $\sqrt{n}/2$. Graphs with such properties can exist only if n is an even square. It is easy to see that these graphs are related to Hadamard matrices, and indeed such relations have been outlined in [4].

As it turns out, if for a graph G equality holds in (1), then its complement $\overline{G}$ is a strongly regular graph which is quite similar but not isomorphic to G, and therefore with $\left\|\overline{G}\right\|_* < (1+\sqrt{n})\, n/2$. This observation led Gutman and Zhou [10] to the following natural question:

What is the maximum $\mathcal{E}(n)$ of the sum $\|G\|_* + \left\|\overline{G}\right\|_*$, where G is a graph of order n?

Questions of this type are the subject matter of this paper. To begin with, note that the Paley graphs, and more generally conference graphs (see below), provide a lower bound

$$\mathcal{E}(n) \geq (n-1)\left(1+\sqrt{n}\right), \tag{2}$$

which closely complements the upper bound

$$\mathcal{E}(n) \leq \sqrt{2}n + (n-1)\sqrt{n-1}$$

proved by Gutman and Zhou in [10]. Although the latter inequality can be improved by a more careful proof (see Theorem 5 below), a gap still remains between the upper and lower bounds on $\mathcal{E}(n)$. In this note we shall close this gap and show that inequality (2) is in fact an equality. This advancement comes at a price of a rather long proof, involving some new analytical and combinatorial techniques based on Weyl's inequalities for sums of Hermitian matrices.

Moreover, we prove our bounds for nonnegative matrices that are more general than adjacency matrices of graphs, and yet the adjacency matrices of conference graphs provide the cases of equality. It is somewhat surprising that purely analytical matrix functions attain their maxima on matrices with a rigid and sophisticated combinatorial structure.

It is also natural to study similar problems for matrix norms different from the trace norm, like the Ky Fan or the Schatten norms. Other parameters can be changed independently and we thus arrive at a whole grid of extremal problems most of which are

open and seem rather difficult. Results of this types and directions for further research are outlined in Section 3.

For reader's sake we start with our graph-theoretic result first.

Theorem 1 *If G is a graph of order $n \geq 7$, then*

$$\|G\|_* + \left\|\overline{G}\right\|_* \leq (n-1)\left(1+\sqrt{n}\right), \tag{3}$$

with equality holding if and only if G is a conference graph.

Recall that a conference graph of order n is a strongly regular graph with parameters

$$(n, (n-1)/2, (n-5)/4, (n-1)/4).$$

It is known that the eigenvalues of a conference graph of order n are

$$(n-1)/2, \left((\sqrt{n}-1)/2\right)^{[(n-1)/2]}, \left(-\left(\sqrt{n}+1\right)/2\right)^{[(n-1)/2]},$$

where the numbers in brackets denote multiplicities. We shall make use of the fact that every graph with these eigenvalues must be a conference graph. Reference material on these questions can be found in [2].

The best known type of conference graphs are the Paley graphs P_q : *Given a prime power* $q = 1 \pmod 4$, *the vertices of* P_q *are the numbers* $1, \ldots, q$ *and two vertices* u, v *are adjacent if* $u - v$ *is an exact square* mod q. The Paley graphs are self complementary, which fits with their extremal property with respect to (3).

As usual, I_n stands for the identity matrix of size n, J_n stands for the all ones matrix of size n, and $\mathbf{j}_n$ is the n-dimensional vector of all ones. Also, $\sigma_1(A) \geq \sigma_2(A) \geq \cdots$ will denote the singular values of a matrix A and $\mu_1(A) \geq \mu_2(A) \geq \cdots$ will denote the eigenvalues of a symmetric matrix A. Finally, $|A|_\infty$ will stand for the maximum of the absolute values of the entries of A.

We shall prove Theorem 1 in the following matrix setup.

Theorem 2 *If A is a symmetric nonnegative matrix of size $n \geq 7$, with $|A|_\infty \leq 1$, and with zero diagonal, then*

$$\|A\|_* + \|J_n - I_n - A\|_* \leq (n-1)\left(1+\sqrt{n}\right), \tag{4}$$

with equality holding if and only if A is the adjacency matrix of a conference graph.

A crucial role in our proof play the following two facts about nonnegative matrices, which are of interest on their own.

Theorem 3 *If A is a square nonnegative matrix of size n, with $|A|_\infty \leq 1$, and with zero diagonal, then*

$$\left\|A + \frac{1}{2}I_n\right\|_* + \left\|J_n - A - \frac{1}{2}I_n\right\|_* \leq n + (n-1)\sqrt{n}. \tag{5}$$

Equality is possible if and only if A is a $(0,1)$-matrix, with all row and column sums equal to $(n-1)/2$, and such that $\sigma_i\left(A + \frac{1}{2}I_n\right) = \sqrt{n}/2$ for $i = 2, \ldots, n$.

Corollary 4 *If A is a symmetric nonnegative matrix of size n, with $|A|_\infty \leq 1$, and with zero diagonal, such that*

$$\left\| A + \frac{1}{2} I_n \right\|_* + \left\| J_n - A - \frac{1}{2} I_n \right\|_* = n + (n-1)\sqrt{n}, \tag{6}$$

then A is the adjacency matrix of a conference graph.

Before proceeding with the proof of Theorem 2 let us note that the difficulty of its proof seems due to the fact that A is a symmetric matrix and has a zero diagonal. In Theorem 5 we shall see that if these conditions are omitted, the proof becomes really straightforward, but unfortunately this result is not tight for graphs.

2 Proofs of Theorems 3 and 2

Our proof of Theorem 3 illustrates the two main ingredients of several proofs later. First, this is an application of the "arithmetic mean-quadratic mean", or the AM-QM inequality. This way we obtain an upper bound on the trace norm by the sum of the squares of the singular values of a matrix, which is equal to the sum of the squares of its entries. Second, we identify a function of the type

$$f(x) = x + \sqrt{a - bx^2}$$

and conclude that $f(x)$ is decreasing in x under certain assumptions about a, b and x. This way we obtain the upper bound $f(x) \leq f(\min x)$.

Proof of Theorem 3 Set for short

$$\begin{aligned} B &= A + \frac{1}{2} I_n, \\ \overline{B} &= J_n - A - \frac{1}{2} I_n. \end{aligned}$$

Applying the AM-QM inequality, we see that

$$\begin{aligned} \|B\|_* + \|\overline{B}\|_* &= \sigma_1(B) + \sigma_1(\overline{B}) + \sum_{i=2}^{n} \sigma_i(B) + \sigma_i(\overline{B}) \\ &\leq \sigma_1(B) + \sigma_1(\overline{B}) + \sqrt{2(n-1)\left(\sum_{i=2}^{n} \sigma_i^2(B) + \sigma_i^2(\overline{B})\right)}. \end{aligned} \tag{7}$$

On the other hand,

$$\sum_{i=1}^{n} \sigma_i^2(B) + \sigma_i^2(\overline{B}) = tr(BB^T) + tr(\overline{B}\,\overline{B}^T) = \sum_{i,j} \left(B_{ij}^2 + \overline{B}_{ij}^2\right) \leq n^2 - \frac{n}{2}, \tag{8}$$

and so

$$\|B\|_* + \left\|\overline{B}\right\|_* \leq \sigma_1(B) + \sigma_1\left(\overline{B}\right) + \sqrt{2(n-1)\left(n^2 - \frac{n}{2} - \sigma_1^2(B) - \sigma_1^2(\overline{B})\right)}$$

$$\leq \sigma_1(B) + \sigma_1\left(\overline{B}\right) + \sqrt{2(n-1)\left(n^2 - \frac{n}{2} - \frac{(\sigma_1(B) + \sigma_1\left(\overline{B}\right))^2}{2}\right)}.$$

Furthermore, using calculus we see that the function

$$f(x) = x + \sqrt{2(n-1)\left(n^2 - \frac{n}{2} - \frac{x^2}{2}\right)}$$

is decreasing in x when $x \geq n$. On the other hand, $\sigma_1(A)$ is the operator norm of A; hence

$$\sigma_1(B) + \sigma_1(\overline{B}) \geq \frac{1}{n}\langle B\mathbf{j}_n, \mathbf{j}_n\rangle + \frac{1}{n}\left\langle \overline{B}\mathbf{j}_n, \mathbf{j}_n\right\rangle = \frac{1}{n}\langle J_n\mathbf{j}_n, \mathbf{j}_n\rangle = n. \tag{9}$$

Thus, $f\left(\sigma_1(B) + \sigma_1\left(\overline{B}\right)\right) \leq f(n)$, and so

$$\|B\|_* + \left\|\overline{B}\right\|_* \leq n + \sqrt{2(n-1)\left(n^2 - \frac{n}{2} - \frac{n^2}{2}\right)},$$

completing the proof of (5).

If equality holds in (5), then we have equality in (7), (8), and (9). Therefore, A is a $(0,1)$-matrix,

$$\sigma_1(B) + \sigma_1\left(\overline{B}\right) = n,$$
$$\sigma_1^2(B) + \sigma_1^2\left(\overline{B}\right) = (\sigma_1(B) + \sigma_1\left(\overline{B}\right))^2/2,$$

and

$$\sigma_2(B) = \cdots = \sigma_n(B) = \sigma_2\left(\overline{B}\right) = \cdots = \sigma_n\left(\overline{B}\right).$$

Hence $\sigma_1(B) = \sigma_1\left(\overline{B}\right) = n/2$ and

$$\sigma_2(B) = \cdots = \sigma_n(B) = \sqrt{n}/2.$$

We omit the simple proof that if $\sigma_1(B) = \frac{1}{n}\langle B\mathbf{j}_n, \mathbf{j}_n\rangle$, then all row and column sums of B are equal, which in our case implies that all row and column sums of A are equal to $(n-1)/2$.

To prove that the fact that if A satisfies the listed conditions, then equality holds in (5) it is enough to check that

$$\sigma_i\left(\overline{B}\right) = \sqrt{n}/2$$

for $i = 2, \ldots, n$. Indeed, since the row and column sums of B are equal to $n/2$, we see that

$$\overline{B}\,\overline{B}^T = (J_n - B)\left(J_n - B^T\right) = nJ_n - \frac{n}{2}J_n - \frac{n}{2}J_n + BB^T = BB^T.$$

and so, $\sigma_i\left(\overline{B}\right)=\sigma_i\left(B\right)=\sqrt{n}/2.$ $\square$

Proof of Corollary 4 From Theorem 3 we know that A is a symmetric $(0,1)$ matrix of a regular graph of degree $(n-1)/2$. Since $\mu_i\left(A+\frac{1}{2}I_n\right)=\pm\sqrt{n}/2$ for $i=2,\ldots,n$, then $\mu_i\left(A\right)=\left(\sqrt{n}-1\right)/2$ or $\mu_i\left(A\right)=-\left(\sqrt{n}+1\right)/2$ for $i=2,\ldots,n$. Using the fact $\mu_1\left(A\right)=(n-1)/2$, we see that A has the spectrum of a conference graph, and therefore A is the adjacency matrix of a conference graph. This completes the proof of Corollary 4. $\square$

Proof of Theorem 2 Our main goal is to prove inequality (4). To keep the proof streamlined we have freed it of a large number of easy, but tedious calculations.

Assume that $n\geq 7$ and let A be a matrix satisfying the conditions of the theorem and such that

$$\left\|A\right\|_*+\left\|J_n-I_n-A\right\|_*$$

is maximal. For short, let

$$\overline{A}=J_n-I_n-A,$$

and set

$$\mu_k=\mu_k\left(A\right),\ \ \overline{\mu}_k=\mu_k\left(\overline{A}\right)$$

for every $k\in[n]$.

We start with the following particular case of Weyl's inequalities: for every $k=2,\ldots,n$,

$$\mu_k+\overline{\mu}_{n-k+2}\leq\mu_2\left(J_n-I_n\right)=-1. \tag{10}$$

Write $n^+\left(A\right)$ for the number of nonnegative eigenvalues of a matrix A. To keep track of the signs of μ_k and $\overline{\mu}_{n-k+2}$, define the set

$$P=\left\{k\mid 2\leq k\leq n,\ \mu_k\geq 0\text{ or }\overline{\mu}_{n-k+2}\geq 0\right\},$$

and let $p=|P|$. Note that if $k\in P$ and $\mu_k\geq 0$, then (10) implies that $\overline{\mu}_{n-k+2}<0$. Thus, in view of $\mu_1\geq 0$ and $\overline{\mu}_1\geq 0$, we see that $n^+\left(A\right)+n^+\left(\overline{A}\right)=p+2$.

The pivotal point of our proof is the value of p, which obviously is at most $n-1$. If $p=n-1$, we shall finish the proof by Theorem 3 and Corollary 4. In the remaining cases, when $p=n-2$ or $p<n-2$, we shall show that strict inequality holds in (4). We note that these two cases require distinct proofs, albeit very similar in spirit.

Let first $p=n-1$. Set for short

$$B=A+\frac{1}{2}I_n,\ \ \overline{B}=J_n-A-\frac{1}{2}I_n=\overline{A}+\frac{1}{2}I_n,$$

and note that

$$\sum_{i=1}^{n}\mu_i(B)=tr\left(B\right)=\frac{n}{2},$$

which implies that

$$\sum_{\mu_i(B)<0}\left|\mu_i(B)\right|=\sum_{\mu_i(B)\geq 0}\mu_i(B)-\frac{n}{2},$$

and by symmetry, also that

$$\sum_{\mu_i(\overline{B})<0} \left|\mu_i(\overline{B})\right| = \sum_{\mu_i(\overline{B})\geq 0} \mu_i(\overline{B}) - \frac{n}{2}.$$

Hence, we find that

$$\|B\|_* + \left\|\overline{B}\right\|_* = \sum_{i=1}^{n} |\mu_i(B)| + \left|\mu_i(\overline{B})\right| = 2\sum_{\mu_i(B)\geq 0} \mu_i(B) + 2\sum_{\mu_i(\overline{B})\geq 0} \mu_i(\overline{B}) - n. \tag{11}$$

On the other hand, we see that

$$\begin{aligned}
\sum_{\mu_i(B)\geq 0} \mu_i(B) + \sum_{\mu_i(\overline{B})\geq 0} \mu_i(\overline{B}) &= \sum_{\mu_i\geq -1/2} |\mu_i + 1/2| + \sum_{\overline{\mu}_i\geq -1/2} |\overline{\mu}_i + 1/2| \\
&\geq \sum_{\mu_i\geq 0} (\mu_i + 1/2) + \sum_{\overline{\mu}_i\geq 0} (\overline{\mu}_i + 1/2) \\
&= \frac{1}{2}\|A\|_* + \frac{1}{2}\left\|\overline{A}\right\|_* + \frac{1}{2}n^+(A) + \frac{1}{2}n^+\left(\overline{A}\right) \\
&= \frac{1}{2}\|A\|_* + \frac{1}{2}\left\|\overline{A}\right\|_* + \frac{1}{2}(p+2)
\end{aligned}$$

Therefore, in view of (11),

$$\|B\|_* + \left\|\overline{B}\right\|_* \geq \|A\|_* + \left\|\overline{A}\right\|_* + p + 2 - n = \|A\|_* + \left\|\overline{A}\right\|_* + 1.$$

and inequality (4) follows by Theorem 3 applied to the matrix B. This completes the proof when $p = n-1$. Note that the characterization of equality in (4) comes directly from Corollary 4, as in the cases when $p = n-2$ or $p < n-2$, a strict inequality always holds in (4).

Let now $p = n-2$. That is to say, there exists exactly one $k \in \{2, \ldots, n\}$ such that $\mu_k < 0$ and $\overline{\mu}_{n-k+2} < 0$. Then, setting

$$\begin{aligned}
x &= \mu_1 + \overline{\mu}_1, \\
y &= |\mu_k| + \left|\overline{\mu}_{n-k+2}\right|,
\end{aligned}$$

we see that $y \geq 1$, and also

$$x = \mu_1 + \overline{\mu}_1 = -\sum_{i=2}^{n} \mu_i + \overline{\mu}_{n-i+2} \geq y + n - 2 \geq n - 1. \tag{12}$$

By the definition of P and Weyl's inequalities (10), for each $i \in P$, we have

$$\mu_i^2 + \overline{\mu}_{n-i+2}^2 = \frac{\left(|\mu_i| + \left|\overline{\mu}_{n-i+2}\right|\right)^2}{2} + \frac{\left(|\mu_i| - \left|\overline{\mu}_{n-i+2}\right|\right)^2}{2} \geq \frac{\left(|\mu_i| + \left|\overline{\mu}_{n-i+2}\right|\right)^2}{2} + \frac{1}{2}.$$

Therefore,

$$\begin{aligned}
n(n-1) &\geq \sum_{i,j} A_{ij}^2 + \overline{A}_{ij}^2 = \sum_{i=1}^{n} \mu_i^2 + \overline{\mu}_i^2 \\
&= \mu_1^2 + \overline{\mu}_1^2 + \mu_k^2 + \overline{\mu}_{n-k+2}^2 + \sum_{i \in P} \mu_i^2 + \overline{\mu}_{n-i+2}^2 \\
&\geq \frac{x^2}{2} + \frac{y^2}{2} + \sum_{i \in P} \frac{\left(|\mu_i| + \left|\overline{\mu}_{n-i+2}\right|\right)^2}{2} + \frac{p}{2} \\
&\geq \frac{x^2}{2} + \frac{y^2}{2} + \frac{1}{2p} \left(\sum_{i \in P} |\mu_i| + \left|\overline{\mu}_{n-i+2}\right| \right)^2 + \frac{p}{2}.
\end{aligned}$$

Replacing p by $n-2$, after some simple algebra we find that

$$\|A\|_* + \left\|\overline{A}\right\|_* \leq x + y + \sqrt{2\,(n-2)\left(n(n-1) - \frac{n-2}{2} - \frac{x^2}{2} - \frac{y^2}{2}\right)}.$$

We shall show that the function

$$f(x,y) = x + y + \sqrt{2(n-2)\left(n(n-1) - \frac{n-2}{2} - \frac{x^2}{2} - \frac{y^2}{2}\right)}$$

is decreasing in x for $x \geq n-1$ and $y \geq 1$. Indeed, otherwise there exist $x \geq n-1$ and $y \geq 1$ such that

$$1 - \frac{(n-2)x}{\sqrt{2(n-2)\left(n(n-1) - \frac{n-2}{2} - \frac{x^2}{2} - \frac{y^2}{2}\right)}} = \frac{\partial f(x,y)}{\partial x} \geq 0,$$

which is a contradiction for $n \geq 5$.

Now, (12) implies that

$$\begin{aligned}
f(x,y) &\leq f(y+n-2, y) \\
&= 2y + n - 2 + \sqrt{(n-2)\left(2n(n-1) - (n-2) - (y+n-2)^2 - y^2\right)}.
\end{aligned}$$

Furthermore, using calculus, we see that $f(y+n-2, y)$ is decreasing in y for $n \geq 6$ and $y \geq 1$. Therefore,

$$f(y+n-2, y) \leq f(n-1, 1) = n + \sqrt{n(n-1)\,(n-2)},$$

and so

$$\|A\|_* + \left\|\overline{A}\right\|_* \leq n + \sqrt{n(n-1)\,(n-2)}.$$

It is not hard to see that for $n \geq 6$,

$$n + \sqrt{n(n-1)\,(n-2)} < n - 1 + (n-1)\sqrt{n},$$

completing the proof of (4) when $p = n - 2$.

Let now $p \leq n - 3$. Then there exist two distinct $k, j \in \{2, \ldots, n\} \setminus P$. Let

$$\begin{aligned} x &= \mu_1 + \overline{\mu}_1, \\ y &= |\mu_k| + \left|\overline{\mu}_{n-k+2}\right| + |\mu_j| + \left|\overline{\mu}_{n-j+2}\right| \end{aligned}$$

From the definition of P and Weyl's inequalities (10) we have

$$y = |\mu_k| + \left|\overline{\mu}_{n-k+2}\right| + |\mu_j| + \left|\overline{\mu}_{n-j+2}\right| \geq 2,$$

and also, by $tr\,(A) = tr\left(\overline{A}\right) = 0$,

$$x = \mu_1 + \overline{\mu}_1 = -\sum_{i=2}^{n} \mu_i + \overline{\mu}_{n-i+2} \geq y + n - 3 \geq n - 1. \tag{13}$$

As in the previous case, we obtain

$$\|A\|_* + \left\|\overline{A}\right\|_* \leq x + y + \sqrt{2(n-3)\left(n(n-1) - \frac{1}{2} - \frac{x^2}{2} - \frac{y^2}{4}\right)}.$$

Next, using calculus, we find that the function

$$f\left(x, y\right) = x + y + \sqrt{2(n-3)\left(n(n-1) - \frac{1}{2} - \frac{x^2}{2} - \frac{y^2}{4}\right)}$$

is decreasing in x for $n \geq 5$, $x \geq n - 1$ and $y \geq 2$. Thus, (13) implies that

$$f(x, y) \leq f(y + n - 3, y).$$

Again using calculus, we find that for $n \geq 7$ and $y \geq 2$, the function

$$f(y + n - 3, y) = 2y + n - 3 + \sqrt{2(n-3)\left(n(n-1) - \frac{1}{2} - \frac{(y+n-3)^2}{2} - \frac{y^2}{4}\right)}$$

is decreasing in y. Therefore,

$$\begin{aligned} \|A\|_* + \left\|\overline{A}\right\|_* &\leq f\left(x, y\right) \leq f(y + n - 3, y) \leq f(n - 1, 2) \\ &= n + 1 + \sqrt{(n-3)\left(n^2 - 4\right)}. \end{aligned}$$

A simple calculation shows that for $n \geq 5$

$$n + 1 + \sqrt{(n-3)\left(n^2 - 4\right)} < n - 1 + (n-1)\sqrt{n},$$

completing the proof of Theorem 2. $\square$

3 Further extensions

An obvious question that arises from the above results is the possibility to extend them to non-symmetric and possibly non-square, nonnegative matrices. We state such extensions in Theorem 5 and 6 below.

Also, the above results focus on the trace norm of matrices, which is known to be a particular case of the Ky Fan k-norms on the one hand, and of the Schatten p-norms on the other. Extremal properties of these norms have been studied in connection to graphs and nonnegative matrices in [7] and [8]. It has been shown that many known results for graph energy carry over these more general norms. Below, the same trend is illustrated for the extremal problems discussed above.

We denote the Ky Fan k-norm of a matrix A by $\|A\|_{*k}$, that is to say

$$\|A\|_{*k} = \sigma_1(A) + \cdots + \sigma_k(A).$$

Also, $J_{m,n}$ will stand for the $m \times n$ matrix of all ones. In what follows we shall focus only on the Ky Fan norms. The Schatten p-norms seem of a slightly different flavor and we shall leave them for further study.

Theorem 5 *If $2 \leq k \leq m \leq n$ and A is a nonnegative matrix of size $m \times n$, with $|A|_\infty \leq 1$, then*

$$\|A\|_{*k} + \|J_{m,n} - A\|_{*k} \leq \sqrt{mn}\left(1 + \sqrt{k-1}\right). \tag{14}$$

Proof Set for short $\overline{A} = J_{m,n} - A$. Following familiar arguments, we see that

$$\begin{aligned}
\|A\|_{*k} + \left\|\overline{A}\right\|_{*k} &= \sigma_1(A) + \sigma_1\left(\overline{A}\right) + \sum_{i=2}^{k}\sigma_i(A) + \sigma_i\left(\overline{A}\right) \\
&\leq \sigma_1(A) + \sigma_1\left(\overline{A}\right) + \sqrt{2(k-1)\left(\sum_{i=2}^{k}\sigma_i^2(A) + \sigma_i^2\left(\overline{A}\right)\right)} \\
&\leq \sigma_1(A) + \sigma_1\left(\overline{A}\right) + \sqrt{2(k-1)\left(mn - \frac{(\sigma_1(A) + \sigma_1\left(\overline{A}\right))^2}{2}\right)}.
\end{aligned}$$

Since the function

$$f(x) = x + \sqrt{2(k-1)\left(mn - \frac{x^2}{2}\right)}$$

is decreasing in x for $x \geq \sqrt{mn}$, and also

$$\sigma_1(A) + \sigma_1(\overline{A}) \geq \frac{1}{\sqrt{mn}}\langle A\mathbf{j}_n, \mathbf{j}_m\rangle + \frac{1}{\sqrt{mn}}\left\langle \overline{A}\mathbf{j}_n, \mathbf{j}_m\right\rangle = \frac{1}{\sqrt{mn}}\langle J_{m,n}\mathbf{j}_n, \mathbf{j}_m\rangle = \sqrt{mn},$$

we see that

$$\|A\|_* + \left\|\overline{A}\right\|_* \leq \sqrt{mn} + \sqrt{(k-1)mn} = \sqrt{mn}\left(1 + \sqrt{k-1}\right),$$

completing the proof of Theorem 5. $\square$

We cannot describe exhaustively the cases of equality in (14). However we shall describe a general construction proving that (14) is exact in a rich set of cases.

Theorem 6 *Let $k \geq 2$ be an integer for which there is a Hadamard matrix of size $k-1$. Let $p, q \geq 1$ be arbitrary integers and set $m = 2p(k-1)$ and $n = 2q(k-1)$. There exists a $(0,1)$-matrix A of size $m \times n$ such that*

$$\|A\|_{*k} + \|J_{m,n} - A\|_{*k} = \sqrt{mn}\left(1 + \sqrt{k-1}\right).$$

Proof Indeed let H be a Hadamard matrix of size $k-1$, and set

$$H' = \begin{pmatrix} H & -H \\ -H & H \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \otimes H,$$

and

$$A = \frac{1}{2}\left((H' \otimes J_{p,q}) + J_{m,n}\right),$$

where $\otimes$ denotes the Kronecker product of matrices. First note that A is a $(0,1)$-matrix of size $m \times n$.

Our main goal is to show that $\sigma_1(A) = \sqrt{mn}/2$ and that

$$\sigma_2(A) = \cdots = \sigma_k(A) = \frac{\sqrt{mn}}{2\sqrt{k-1}}.$$

It is known that H has $k-1$ singular values which are equal to $\sqrt{k-1}$. Therefore H' has $k-1$ singular values all equal to $2\sqrt{k-1}$. Next, we see that $H' \otimes J_{p,q}$ has $k-1$ nonzero singular values, all equal to $2\sqrt{pq(k-1)}$. We also see that the row and column sums of H' are 0 and thus 0 is a singular value of H' with singular vectors $\mathbf{j}_{2(k-1)}, \mathbf{j}_{2(k-1)}$; hence, 0 is a singular value of $H' \otimes J_{p,q}$ with singular vectors $\mathbf{j}_m$ and $\mathbf{j}_n$. Since $\mathbf{j}_m$ and $\mathbf{j}_n$ are also singular vectors to the unique nonzero singular value of $J_{m,n}$, it is easy to see that the singular values of A are exactly as described above.

Hence

$$\|A\|_{*k} = \frac{\sqrt{mn}}{2} + \frac{\sqrt{mn}}{2\sqrt{k-1}}(k-1) = \frac{\sqrt{mn}}{2}\left(1 + \sqrt{k-1}\right).$$

On the other hand

$$J_{m,n} - A = \frac{1}{2}\left((-H' \otimes J_{p,q}) + J_{m,n}\right)$$

and so the $J_{m,n} - A$ has the same singular values as A, since $-H$ is a Hadamard matrix as well. This complete the proof of Theorem 6. $\square$

Two open problems arise in connection to Theorems 2, 5 and 6.

Problem 7 *Describe all matrices A for which equality holds in (14).*

Problem 8 *Let A be a symmetric nonnegative matrix of size n, with $|A|_\infty \leq 1$, with zero diagonal. Find the maximum of*

$$\|A\|_{*k} + \|J_n - I_n - A\|_{*k}.$$

It seems very likely that the latter problem can be solved along the lines of Theorem 3.

Note that we have stated and proved Theorem 5 for $k \geq 2$. As it turns out the important case $k = 1$ (operator norm) is indeed particular.

Theorem 9 *If A is an* $m \times n$ nonnegative matrix, with $|A|_\infty \leq 1$, then,

$$\sigma_1(A) + \sigma_1(J_{m,n} - A) \leq \sqrt{2mn}, \tag{15}$$

with equality holding if and only if mn is even, and A is a $(0,1)$-matrix with precisely $mn/2$ ones that are contained either in $n/2$ columns or in $m/2$ rows.

Proof We have

$$\sigma_1^2(A) + \sigma_1^2(J_{m,n} - A) \leq \sum_{i=1}^{m} \sigma_i^2(A) + \sigma_i^2(J_{m,n} - A) = \sum_{ij} A_{ij}^2 + (1 - A_{ij})^2 \leq mn, \tag{16}$$

and inequality (15) follows by the the AM-QM inequality.

If equality holds in (15), then we have equalities throughout (16). Therefore, A is a $(0,1)$-matrix of rank 1, and has precisely $mn/2$ ones. Thus, the ones of A form a submatrix B of A, Since $J_{m,n} - A$ is also a rank 1 matrix, its ones are also contained in a submatrix B' of $J_{m,n} - A$. Then B and B' are submatrices of $J_{m,n}$ that do not share entries and together contain all entries of $J_{m,n}$. Clearly B must be of size either $m/2 \times n$ or $m \times n/2$. This completes the proof of Theorem 9. $\square$

In the spirit of the previous problems, one can ask the following question: *Let A be a symmetric nonnegative matrix of size n, with $|A|_\infty \leq 1$, and with zero diagonal. What is the maximum of*

$$\mu_1(A) + \mu_1(J_n - I_n - A).$$

This question turns to be known and difficult, but it has been recently answered in [1] and [9].

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